Study of chaos in hamiltonian systems via convergent normal forms

Werner M. Vieira†, Alfredo M. O. de Almeida‡

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Abstract

We use Moser's normal forms to study chaotic motion in two-degree hamiltonian systems near a saddle point. Besides being convergent, they provide a suitable description of the cylindrical topology of the chaotic flow in that vicinity. Both aspects combined allowed a precise computation of the homoclinic interaction of stable and unstable manifolds in the full phase space, rather than just the Poincaré section. The formalism was applied to the Hénon-Heiles hamiltonian, producing strong evidence that the region of convergence of these normal forms extends over that originally established by Moser.

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†Departamento de Matemática Aplicada, Universidade Estadual de Campinas, CP 6065, 13083-970 Campinas, SP, Brazil. e-mail: vieira@ime.unicamp.br

‡Centro Brasileiro de Pesquisas Físicas, Rua Xavier Sigaud, 150, 2290-180 Rio de Janeiro, RJ, Brazil.

I. INTRODUCTION

Normal Forms (NF) are among the successful methods for either analytic or numeric studies of dynamical systems; by performing a suitable coordinate transformation, we eventually obtain a more simple or dynamically "transparent" version of the original system. This approach can be formulated either for generic systems including two-dimensional maps [1] or for hamiltonian systems [2] and conservative maps [3]. So, even when a given NF does not converge, due to small denominators or exact ressonances, it is of importance for numerical purposes. This is just what occurs when the NF is obtained around a stable point or orbit. Then, in spite of the well known divergence in this case, a truncation allows us to follow the motion for a long time with great precision. In fact, the nonconvergent case is the most considered in the literature [4–6].

The present work concerns the NF around a *unstable* point or orbit of a conservative system. For that, Moser demonstrated their convergence for both maps [7] and hamiltonian systems [8]. Although convergent, this case did not receive much attention until recently, presumably because the particle remains a very short time in that region. Nevertheless, we will see that the Moser normal forms (MNF) are both convergent and a powerfull tool for the search for the basic structures of the chaotic motion rather than just following a specific orbit for a long time.

The usefulness of the MNF for the study of conservative chaotic maps is already known in the literature. It allowed precise analytical computations of homoclinic points and the periodic points with long period, which accumulate in the homoclinic ones [9,10]. Additional good numerical results were obtained even if small dissipative perturbations were added [11].

Area-preserving maps are usually only simplified reductions (appropriate Poincaré sections) of autonomous hamiltonian systems of two degrees of freedom. In particular, the very complex two-dimensional homoclinic tangle [12] is already a reduction of the much more involved chaotic motion in the full phase space. However, extending the use of the MNF to the hamiltonian case allows us to study directly the proper four-dimensional chaotic flow.

This has previously been attempted in the literature [13] , but without taking full advantage of the method that we shall develop here. Another use of MNFs is found in [14] and takes advantage of its convergence to ascertain stability transitions of families of periodic orbits near hamiltonians' saddle points.

In section II we will develop the MNF approach for the case of a generic autonomous hamiltonian of two degrees of freedom around a saddle point, encompassed by Moser's convergence proof. In that vicinity, the flow's topology is *cylindrical* rather than toroidal, in the case of a chaotic regime [15]. Firstly, we construct the relations linking the original system to the corresponding more "transparent" normalized system. This is done through a near identity polynomial coordinate transformation. We will see that, besides convergent, that transformation also reveals in a natural way, the cylindrical character of the topology. Both features turn the MNF into a powerfull tool. So, it was possible to compute, precisely for the first time, the *continuous* structures in the full phase space, underlying the homoclinic tangle in a Poincaré section: the (un)stable manifolds which originate at the saddle point and at each neighbouring unstable periodic orbit, the homoclinic orbits associated with the latter and the periodic orbits with long period which accumulate on the homoclinic orbits. In section III we obtain the recurrence relations for the coefficients involved in the theory.

In sections IV and V we apply the formalism to the specific case of the Hénon-Heiles hamiltonian. The numerical results exhibited in section V fully confirmed the expectations about the MNF as a tool for the study and characterization of chaotic motions. Moreover, they also point to some kind of extension of the region of convergence initially assumed for Moser's theorem. In fact, this issue is just being considered by the authors presently.

Finally, in section VI we summarize the results and possible extensions of the present work.

II. MOSER'S NORMAL FORM

It is essential that the hamiltonian be in the complexified form, for the method's implementation:

$$h(x_1, x_3, x_2, x_4) \;=\; \lambda_1 x_1 x_3 + \lambda_2 x_2 x_4 + \sum_{\ell=3}^{\infty} H(\underline{\ell})\, \underline{x}^{\underline{\ell}} \quad . \tag{1}$$

Here, the origin is assumed to be a saddle point and we use the notation: $\underline{\ell} = (\ell_1, \ell_3, \ell_2, \ell_4)$, $\underline{x} = (x_1, x_3, x_2, x_4)$, $\ell = \ell_1 + \ell_3 + \ell_2 + \ell_4$ and $\underline{x}^{\underline{\ell}} = x_1^{\ell_1} x_3^{\ell_3} x_2^{\ell_2} x_4^{\ell_4}$. The positions are x_1 and x_2 and the conjugate momenta are, respectively, x_3 and x_4. The eigenvalues of the system's linear part are $\lambda_1 = i\omega$ and $\lambda_2 = -\lambda$, with ω and λ real (the other two being of course $-i\omega$ and λ). $H(\underline{\ell})$ is the coefficient of $\underline{x}^{\underline{\ell}}$ and ℓ is its order.

The usual noncomplexified form of the quadratic part h_2 of (1), around the saddle point, is

$$h_2(q_1, q_2, p_1, p_2) \;=\; \frac{\lambda}{2}(p_2^2 - q_2^2) + \frac{\omega}{2}(p_1^2 + q_1^2) \quad , \tag{2}$$

both sets of coordinates being linked by the symplectic transformation

$$\begin{cases} q_1 \;=\; \frac{1}{\sqrt{2}}(x_1 + i x_3) \\ p_1 \;=\; \frac{1}{\sqrt{2}}(i x_1 + x_3) \\ q_2 \;=\; \frac{1}{\sqrt{2}}(x_2 + x_4) \\ p_2 \;=\; \frac{1}{\sqrt{2}}(-x_2 + x_4) \quad . \end{cases} \tag{3}$$

The Hamilton equations coming from (1) are:

$$\begin{cases} \dot{x}_1 \;=\; \lambda_1 x_1 + \sum_{\ell=3}^{\infty} \ell_3\, H(\underline{\ell})\, x_1^{\ell_1} x_3^{\ell_3 - 1} x_2^{\ell_2} x_4^{\ell_4} \\ \dot{x}_3 \;=\; -\lambda_1 x_3 - \sum_{\ell=3}^{\infty} \ell_1\, H(\underline{\ell})\, x_1^{\ell_1 - 1} x_3^{\ell_3} x_2^{\ell_2} x_4^{\ell_4} \\ \dot{x}_2 \;=\; \lambda_2 x_2 + \sum_{\ell=3}^{\infty} \ell_4\, H(\underline{\ell})\, x_1^{\ell_1} x_3^{\ell_3} x_2^{\ell_2} x_4^{\ell_4 - 1} \\ \dot{x}_4 \;=\; -\lambda_2 x_4 - \sum_{\ell=3}^{\infty} \ell_2\, H(\underline{\ell})\, x_1^{\ell_1} x_3^{\ell_3} x_2^{\ell_2 - 1} x_4^{\ell_4} \quad . \end{cases} \tag{4}$$

Moser's theorem assures that there exists a near-identity polynomial coordinate transformation, convergent in a neighbourhood of the origin,

$$x_i \;=\; y_i + \sum_{\ell=2}^{\infty} X(i,\underline{\ell})\,\underline{y}^{\underline{\ell}} \quad , \qquad\qquad i \;=\; 1,2,3,4 \quad , \tag{5}$$

such that under it the original system (4) takes the *normalized* form

$$\begin{cases} \dot{y}_1 \;=\; F(y_1y_3\,,\, y_2y_4)\, y_1 \\ \dot{y}_3 \;=\; -F(y_1y_3\,,\, y_2y_4)\, y_3 \\ \dot{y}_2 \;=\; G(y_1y_3\,,\, y_2y_4)\, y_2 \\ \dot{y}_4 \;=\; -G(y_1y_3\,,\, y_2y_4)\, y_4 \quad , \end{cases} \tag{6}$$

the functions F and G depending as indicated only on the products y_1y_3 and y_2y_4. In fact, the normal system (6) is also hamiltonian, with Hamilton function $k(\underline{y})$ given by

$$k(y_1,y_3,y_2,y_4) \;=\; \lambda_1 y_1y_3 + \lambda_2 y_2y_4 + \sum_{m=2}^{\infty} K(2\underline{m})\,(y_1y_3)^{m_1}(y_2y_4)^{m_2} \quad , \tag{7}$$

where $K(2\underline{m}) = K(2m_1, 2m_2)$ are the expansion's coefficients.

It is straightforward to see that the two constants of motion of (6) are just the products

$$y_1y_3 \;=\; c_1 \quad , \tag{8}$$

$$y_2y_4 \;=\; c_2 \quad , \tag{9}$$

in terms of which the system can immediately be integrated:

$$\begin{cases} y_1(t) \;=\; y_1(0)\, \exp\,(t\,F) \\ y_3(t) \;=\; y_3(0)\, \exp\,(-\,t\,F) \\ y_2(t) \;=\; y_2(0)\, \exp\,(t\,G) \\ y_4(t) \;=\; y_4(0)\, \exp\,(-\,t\,G) \quad , \end{cases} \tag{10}$$

with F and G given by

$$\begin{cases} F(y_1y_3, y_2y_4) \;=\; \lambda_1 + \sum_{m=2}^{\infty} m_1\, K(2\underline{m})\,(y_1y_3)^{m_1-1}(y_2y_4)^{m_2} \\ G(y_1y_3, y_2y_4) \;=\; \lambda_2 + \sum_{m=2}^{\infty} m_2\, K(2\underline{m})\,(y_1y_3)^{m_1}(y_2y_4)^{m_2-1} \quad . \end{cases} \tag{11}$$

It is important to note that the coordinate transformation 5 need not to be canonical, even though the transformed system is also hamiltonian. The cylindrical topology of the

flow can be made explicit. As the original coordinates q_1, q_2, p_1, p_2 are real, we see from (3) that x_2 and x_4 are real and x_1 and x_3 are complex, there existing the following relation between x_1 and x_3:

$$x_3 \; = \; -i\,\overline{x}_1 \quad . \tag{12}$$

The overline symbol above indicates the complex conjugate operation. Now, the first order truncation of (5) permits us to extend the above conclusions to the corresponding coordinates $\underline{y}$, namely y_2 and y_4 are real and y_1 and y_3 are complex satisfying the relations

$$y_3 \; = \; -i\,\overline{y}_1 \quad , \quad |y_1| = |y_3| \equiv \rho \quad , \tag{13}$$

$$y_1 y_3 \; = \; -i\,\rho^2 \quad , \tag{14}$$

ρ being a new constant of motion. Then, it follows from (10) that F is pure imaginary and G is real. If we define (remembering that $\lambda_1 = i\omega$ and $\lambda_2 = -\lambda$)

$$\begin{cases} \Omega \;\equiv\; F/i \;=\; \omega + \delta\omega \\ \Lambda \;\equiv\; G \;=\; \lambda + \delta\lambda \quad , \end{cases} \tag{15}$$

and

$$\begin{cases} \delta\omega \;\equiv\; -i \sum_{m=2}^{\infty} m_1\, K(2\underline{m})\, (-i\rho^2)^{m_1-1} (\epsilon)^{m_2} \\ \delta\lambda \;\equiv\; \sum_{m=2}^{\infty} m_2\, K(2\underline{m})\, (-i\rho^2)^{m_1} (\epsilon)^{m_2-1} \quad , \end{cases} \tag{16}$$

we can rewrite (10) as

$$\begin{cases} y_1(t) \;=\; \rho\, \exp\left[i(\theta_0 + \Omega\, t)\right] \\ y_3(t) \;=\; \rho\, \exp\left[-i(\theta_0 + \frac{\pi}{2} + \Omega\, t)\right] \\ y_2(t) \;=\; y_{20}\, \exp\left[\Lambda\, t\right] \\ y_4(t) \;=\; y_{40}\, \exp\left[-\Lambda\, t\right] \quad , \quad y_2 y_4 \;=\; \epsilon \quad , \end{cases} \tag{17}$$

so that ϵ is the second constant of motion. The convergence of the MNF guarantees that ρ and ϵ are proper constants of the motion. The solution (17) has an obvious cylindrical character: each given orbit, characterized by the phase θ_0, slides on the cylinder that is the

direct product of a circle with radius ρ in the plane y_1, y_3, with a hyperbola with parameter ϵ in the plane y_2, y_4. Obviously, this topology is preserved in the original coordinates, due to the convergence of (5). For a linear flow generated just by h_2 in (2) all frequencies Ω and velocities Λ degenerate respectively into ω and λ, no matter what particular cylinder is considered.

The suitable form of the solution (17) immediately reveals some important structures near the saddle point. Let us consider $\rho = 0$, in which case the motion is confined to the plane y_2, y_4. For $\epsilon = 0$, we identify the coordinate axes y_2 and y_4 as just the stable and unstable orbits originating from the saddle point. For $\epsilon \neq 0$, we have hyperbolae near the axes.

Consider now the case $\rho \neq 0$ (see figure 1). If $y_2 = y_4 = 0$ ($\epsilon = 0$), we have the family of unstable (circular) periodic orbits confined to the plane y_1, y_3. Let τ be one such orbit. If now only $y_4 = 0$, then the motion is composed by τ in that plane and the axis y_2, thus generating from τ the two unstable cylinders (U_1 and U_2). On the other hand, if only $y_2 = 0$, the composition of τ with the axis y_4 generates the pair of stable semicylinders (S_1 and S_2). Now, let us intersect any of the four semicylinders above with a plane transverse to it. As a consequence of the Poincaré-Cartan theorem [16], the symplectic area or action of the closed irreducible curve, formed at the intersection, will equal that of τ. In fact, this is valid for any irreducible curve over the semicylinder. On the other hand, it is a lagrangian surface i.e. all reducible curves on it have null action. Finally, if we compose a circular motion in the plane y_1, y_3 with a hyperbola in the plane y_2, y_4 ($\epsilon \neq 0$), we do not have semicylinders but smooth cylinders near the saddle point, distinct from those associated to the orbits τ.

Of course, the "rectified" structures described above in the $\underline{y}$ coordinates, will appear distorted — yet preserving their topology — in the original coordinates. Moreover, the distorsions are such that, far away from the saddle point, the cylinders execute a very complicated tangle among themselves if the regime is chaotic [15]. Because they are all confined to a compact energy surface in phase space, transverse crossings of such cylinders eventually occur far from the origin. The existence of those transverse crossings are the

source of chaotic motion in the continuum flow. So, an orbit at a transverse intersection of an unstable semicylinder with a stable one (U_1 with S_2 or U_2 with S_1 in the discussion above) is just a homoclinic orbit which tends to the respective orbit τ as $t \to \pm\infty$. In the case of the families of smooth cylinders in the neighbourhood of (U_1, S_2) or (U_2, S_1), the selfcrossings eventually lead to closed orbits with increasing periods, which accumulate on the homoclinic orbits. All these continuum structures underly the homoclinic tangle observed in a (convenient, yet generally nonplanar) Poincaré section of the flow.

Because of its convergence, the MNF formalism allows us to start from precise initial conditions near the saddle point. Moreover, the preceding discussion shows the necessity of the MNF for the general definition of a given cylinder in phase space. This permits us to extend the (un)stable cylinders beyond the vicinity of the origin [15] and proceed to an accurate quantitative study of the structures presented above.

III. RECURRENCE RELATIONS

To consistently determine the coefficients $X(i, \underline{\ell})$ of (5) and $K(2\underline{m})$ of (7), we insert (5) in (4) and compare the result with the system directly obtained from the hamiltonian (7). We see that this task, simple in principle, is laborious if we are interested in an algorithm to compute the coefficents $X(i, \underline{\ell})$ and $K(2\underline{m})$, up to an arbitrary order. Anyway, the recurrence relations we obtain are the following:

$$\begin{aligned}
&A(1,\underline{n})\,X(1,\underline{n}) + B(1,\underline{n})\,K(2n_1, 2n_2) \ = \\
&(n_3+1)\,H(n_1, n_3+1, n_2, n_4) + \\
&\Theta(n-3)\sum_{\ell=2}^{n-1}(\ell_3+1)\,H(\ell_1, \ell_3+1, \ell_2, \ell_4)\,Z^{\underline{\ell}}_{\underline{n}} - \\
&\Theta(n-4)\sum_{m=2}^{INT(n/2)} K(2\underline{m})\,W^{\underline{m}}_{1,\underline{n}} \quad ,
\end{aligned} \tag{18}$$

$$
\begin{aligned}
&A(3,\underline{n})\,X(3,\underline{n}) + B(3,\underline{n})\,K(2n_3,2n_2) \ = \\
&-(n_1+1)\,H(n_1+1,n_3,n_2,n_4) - \\
&\Theta(n-3)\sum_{\ell=2}^{n-1}(\ell_1+1)\,H(\ell_1+1,\ell_3,\ell_2,\ell_4)\,Z_{\underline{n}}^{\underline{\ell}} - \\
&\Theta(n-4)\sum_{m=2}^{INT(n/2)} K(2\underline{m})\,W_{3,\underline{n}}^{\underline{m}} \quad ,
\end{aligned} \tag{19}
$$

$$
\begin{aligned}
&A(2,\underline{n})\,X(2,\underline{n}) + B(2,\underline{n})\,K(2n_1,2n_2) \ = \\
&(n_4+1)\,H(n_1,n_3,n_2,n_4+1) + \\
&\Theta(n-3)\sum_{\ell=2}^{n-1}(\ell_4+1)\,H(\ell_1,\ell_3,\ell_2,\ell_4+1)\,Z_{\underline{n}}^{\underline{\ell}} - \\
&\Theta(n-4)\sum_{m=2}^{INT(n/2)} K(2\underline{m})\,W_{2,\underline{n}}^{\underline{m}} \quad ,
\end{aligned} \tag{20}
$$

$$
\begin{aligned}
&A(4,\underline{n})\,X(4,\underline{n}) + B(4,\underline{n})\,K(2n_1,2n_4) \ = \\
&-(n_2+1)\,H(n_1,n_3,n_2+1,n_4) - \\
&\Theta(n-3)\sum_{\ell=2}^{n-1}(\ell_2+1)\,H(\ell_1,\ell_3,\ell_2+1,\ell_4)\,Z_{\underline{n}}^{\underline{\ell}} - \\
&\Theta(n-4)\sum_{m=2}^{INT(n/2)} K(2\underline{m})\,W_{4,\underline{n}}^{\underline{m}} \quad .
\end{aligned} \tag{21}
$$

The definitions involved here are:

$$
INT(x) \ = \ integer\ part\ of\ x \quad ,
$$

$$
\Theta(x) \ = \ 1\ if\ x \geq 0 \ \ or \ \ 0\ if\ x < 0 \quad ,
$$

$$
\begin{cases}
A(1,\underline{n}) \ = \ \lambda_1(n_1-n_3-1)+\lambda_2(n_2-n_4) \\
A(3,\underline{n}) \ = \ \lambda_1(n_1-n_3+1)+\lambda_2(n_2-n_4) \\
A(2,\underline{n}) \ = \ \lambda_1(n_1-n_3)+\lambda_2(n_2-n_4-1) \\
A(4,\underline{n}) \ = \ \lambda_1(n_1-n_3)+\lambda_2(n_2-n_4+1) \quad ,
\end{cases} \tag{22}
$$

$$\begin{cases} B(1,\underline{n}) &= n_1 \delta_{n_3}^{n_1-1} \delta_{n_4}^{n_2} \\ B(3,\underline{n}) &= -n_3 \delta_{n_3}^{n_1+1} \delta_{n_4}^{n_2} \\ B(2,\underline{n}) &= n_2 \delta_{n_3}^{n_1} \delta_{n_4}^{n_2-1} \\ B(4,\underline{n}) &= -n_4 \delta_{n_3}^{n_1} \delta_{n_4}^{n_2+1} \quad , \end{cases} \tag{23}$$

$$\delta_j^i \;=\; 1 \; if \; i=j \;\; or \;\; 0 \; if \; i \neq j \quad ,$$

$$\begin{aligned} W_{i,\underline{n}}^{\underline{m}} \;=\; & m_1(n_1-n_3)\, X(i, n_1-m_1+1, n_3-m_1+1, n_2-m_2, n_4-m_2)\times \\ & \times\Theta(n_1-m_1+1)\,\Theta(n_3-m_1+1)\,\Theta(n_2-m_2)\,\Theta(n_4-m_2) + \\ & m_2(n_2-n_4)\, X(i, n_1-m_1, n_3-m_1, n_2-m_2+1, n_4-m_2+1) \times \\ & \times\Theta(n_1-m_1)\,\Theta(n_3-m_1)\,\Theta(n_2-m_2+1)\,\Theta(n_4-m_2+1) \quad . \end{aligned} \tag{24}$$

Given $\underline{n}$ and $\underline{m}$, the coefficients $W_{i,\underline{n}}^{\underline{m}}$ depend only on the $X(i,\underline{k})$ of order $k = n-2m+2$. As the minimum value of m is 2, it follows that $k \leq n-2$ and hence $k < n$. As the maximum value of $\underline{m}$ is $INT(n/2)$, it also follows that $k \geq 2$.

The coefficients $Z_{\underline{n}}^{\underline{\ell}}$ arise through the relations

$$\begin{aligned} \underline{x}^{\underline{\ell}} \;&=\; [x_1(\underline{y})]^{\ell_1}\,[x_3(\underline{y})]^{\ell_3}\,[x_2(\underline{y})]^{\ell_2}\,[x_4(\underline{y})]^{\ell_4} \\ &=\; [y_1 + \sum_{i=2}^{\infty} X(1,\underline{i})\,\underline{y}^{\underline{i}}]^{\ell_1}\,[y_3 + \sum_{j=2}^{\infty} X(3,\underline{j})\,\underline{y}^{\underline{j}}]^{\ell_3} \times \\ & \times[y_2 + \sum_{k=2}^{\infty} X(2,\underline{k})\,\underline{y}^{\underline{k}}]^{\ell_2}\,[y_4 + \sum_{m=2}^{\infty} X(4,\underline{m})\,\underline{y}^{\underline{m}}]^{\ell_4} \\ &\equiv\; \underline{y}^{\underline{\ell}} + \sum_{n=\ell+1}^{\infty} Z_{\underline{n}}^{\underline{\ell}}\,\underline{y}^{\underline{n}} \quad . \end{aligned} \tag{25}$$

Obtaining the coefficients $Z_{\underline{n}}^{\underline{\ell}}$ is the heaviest computational task of the whole algorithm. The reason, simply stated, is that we have to obtain a series resulting from multiplying four terms, each of them being itself a series powered to an arbitrary integer! Given an index vector $\underline{\ell}$, the coefficients $Z_{\underline{n}}^{\underline{\ell}}$ depend only on the $X(i,\underline{k})$ with $k < n$. To obtain the $Z_{\underline{n}}^{\underline{\ell}}$ up to order $n = N$, the series in (25) needs to be truncated at combined minimal orders which depend on N. If N changes (increases), these series' minimal truncations also change,

modifying the values of various $Z^{\ell}_{\underline{n}}$ of order $n \le N$ previously obtained. Then, for each increase of N, we need to compute *all* $Z^{\ell}_{\underline{n}}$ again, since we do not know (at least at the present stage) what coefficients, coming from the previous step, will not be changed by the actual calculation. Nevertheless, the process evidently converges as $N \to \infty$.

Now, let us consider a subset of coefficients $Z^{\ell}_{\underline{n}}$ characterized by having some null indexes in $\underline{n} = (n_1, n_3, n_2, n_4)$. It is easy to see that this subset depends only on the $X(i, \underline{k})$ with $\underline{k}$ having the same null structure as $\underline{n}$. Then, the more null indexes the vector $\underline{n}$ has, the easier we can compute the corresponding subset $Z^{\ell}_{\underline{n}}$ from the smaller number of coefficients of the respective subset $X(i, \underline{k})$. Now, let us return to the cylindrical structures discussed in the end of the last section. We need only the coefficients $X(i, n_1, n_3, 0, 0)$ or $X(i, 0, 0, n_2, n_4)$ to respectively describe, in the original coordinates, the normalized planes $y_2 = y_4 = 0$ or $y_1 = y_3 = 0$. Unfortunately, the coefficients $X(i, n_1, n_3, 0, 0)$ and $X(i, 0, 0, n_2, n_4)$ depend by themselves on the remaining ones, through their relations with the coefficients $K(2\underline{m})$. So, we find that the decoupling occurs only if just one index is nonnull, as for e. g. the subsets $X(i, 0, 0, n_2, 0)$ or $X(i, 0, 0, 0, n_4)$. In particular, the later coefficients are just those we need to compute the (un)stable manifolds originated at the saddle point. In that case, the corresponding coefficients $Z^{\ell}_{\underline{n}}$ can be obtained up to a much higher order than the full set, as we will see in section V.

Let us check the consistency of the recurrence relations (18) to (21). Each first member is a linear combination of the unknowns $X(i, \underline{n})$ and $K(2\underline{m})$, and the second members depend on the known hamiltonian coefficients $H(\underline{\ell})$, on the $X(i, \underline{n})$ through $Z^{\ell}_{\underline{n}}$ and $W^{m}_{i,\underline{n}}$ and explicitly on the $K(2\underline{m})$. Nevertheless, the appearance of $X(i, \underline{n})$ and $K(2\underline{m})$ in the second members occurs at lower orders than in the corresponding first ones, as expected. Additionally, they have the following property: if $A(i, \underline{n}) \neq 0$ then $B(i, \underline{n}) = 0$ and reciprocally. This is due to the eigenvalues λ_1 and λ_2 being independent over the reals. So, those relations naturally split into two cases: firstly, we have $A(i, \underline{n}) \neq 0$ and $B(i, \underline{n}) = 0$. This case allows us to obtain the corresponding $X(i, \underline{n})$ in a direct way.

The case $A(i, \underline{n}) = 0$ and $B(i, \underline{n}) \neq 0$ can only occurs for odd orders n. This case

has redundancies so, in order to obtain all the $K(2\underline{m})$, it is only necessary to work with either the pair of eqs. (18) and (20) or (19) and (21). The coefficients $X(i, \underline{n})$ that remain undetermined in this case are the following: $X(1, k_1+1, k_1, k_2, k_2)$, $X(3, k_1, k_1+1, k_2, k_2)$, $X(2, k_1, k_1, k_2+1, k_2)$ and $X(4, k_1, k_1, k_2, k_2+1)$, with $k_1+k_2 = 1, 2, 3, \cdots$. In fact, the Moser theorem imposes some additional normalizing conditions to these "weak" coefficients, which can be simply met by requiring that all of them vanish. This will be assumed here. It should be noted that requiring the coordinate transformation to be canonical reduces this freedom and may not be compatible with the conditions of Moser's theorem.

IV. PREPARATION OF THE HÉNON-HEILES HAMILTONIAN

We apply the preceding formalism to the Hénon-Heiles hamiltonian:

$$h(q_1, q_2, p_1, p_2) = \frac{1}{2}(p_1^2 + p_2^2) + U(q_1, q_2) \quad , \tag{26}$$

$$U(q_1, q_2) = \frac{1}{2}(q_1^2 + q_2^2 + 2q_1^2 q_2 - \frac{2}{3}q_2^3) \quad . \tag{27}$$

Here, q_1 and q_2 are the positions and p_1 and p_2 are their respective momenta.

Even though the motivation for the hamiltonian (26) was astronomical, it soon became relevant on its own, due to both its simplicity and its dynamical richness [4,5,12,17]. For, if the energy $h \equiv E$ is suficiently small, the chaotic regions of the flow are vanishingly small. On the other hand, if the energy increases, the chaotic regions arbitrarily increase too. Additionally, (26) comes from the truncation of the Toda hamiltonian (see e. g. [12]), the later one being totally integrable for all energies.

In figure 2 we exhibit the equipotential curves of (27). The origin $P_0(q_1 = 0, p_1 = 0, q_2 = 0, p_2 = 0)$ is an eliptic (stable) equilibrium point with the eigenvalues $\pm i$, while $P_1(0, 0, 1, 0)$, $P_2(-\sqrt{3}/2, 0, -1/2, 0)$ and $P_3(\sqrt{3}/2, 0, -1/2, 0)$ are saddle (unstable) points, all having the same set of eigenvalues $\pm i\sqrt{3}, \pm 1$. These saddle points are just as required by our method. The evident trigonal symmetry of figure 2 makes them mutually equivalent from a dynamical point of view. Then, there is just one MNF valid around P_1, P_2 and P_3.

To put the hamiltonian (26) in the suitable form (1), we need to perform the following coordinate transformations for P_j , $j = 1, 2, 3$:

$$\begin{pmatrix} q_1 \\ p_1 \\ q_2 \\ p_2 \end{pmatrix} = \begin{pmatrix} \tilde{q}_1 \\ \tilde{p}_1 \\ \tilde{q}_2 \\ \tilde{p}_2 \end{pmatrix} + P_j^T \quad , \tag{28}$$

$$\begin{pmatrix} \tilde{q}_1 \\ \tilde{p}_1 \\ \tilde{q}_2 \\ \tilde{p}_2 \end{pmatrix} = \underline{A}_j \begin{pmatrix} \tilde{z}_2 \\ \tilde{z}_1 \\ \tilde{z}_4 \\ \tilde{z}_3 \end{pmatrix} \quad , \tag{29}$$

$$\begin{pmatrix} \tilde{z}_1 \\ \tilde{z}_3 \\ \tilde{z}_2 \\ \tilde{z}_4 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt[4]{3}} & 0 & 0 & 0 \\ 0 & \sqrt[4]{3} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_3 \\ z_2 \\ z_4 \end{pmatrix} \quad , \tag{30}$$

$$\begin{pmatrix} z_1 \\ z_3 \\ z_2 \\ z_4 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i & 0 & 0 \\ i & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_3 \\ x_2 \\ x_4 \end{pmatrix} \quad . \tag{31}$$

Here, P_j^T is the transpose of the respective saddle points given in the text and the matrices $\underline{A}_j$ are given by

$$\underline{A}_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{32}$$

if the point is P_1, and

$$\underline{A}_j = \begin{pmatrix} cos\theta_j & sin\theta_j & 0 & 0 \\ -sin\theta_j & cos\theta_j & 0 & 0 \\ 0 & 0 & cos\theta_j & sin\theta_j \\ 0 & 0 & -sin\theta_j & cos\theta_j \end{pmatrix} , \quad j = 2, 3 \quad , \tag{33}$$

if the point is P_2 or P_3, with $\theta_2 = \frac{5\pi}{6}$ and $\theta_3 = \frac{\pi}{6}$. All transformations made here are symplectic.

In the intermediate $\underline{z}$ coordinates, the hamiltonian form is

$$h(\underline{z}) = \frac{1}{2}(z_4^2 - z_2^2) + \frac{\sqrt{3}}{2}(z_3^2 + z_1^2) + \frac{\sqrt{3}}{3} z_1^2 z_2 - \frac{1}{3} z_2^3 + \frac{1}{6} \quad . \tag{34}$$

The dissociation energy $E = 1/6$ arises explicitly here. The final hamiltonian form, in the $\underline{x}$ coordinates, is

$$h(x_1, x_3, x_2, x_4) = i\sqrt{3} x_1 x_3 - x_2 x_4 + \frac{\sqrt{6}}{12}(x_1 + i x_3)^2 (x_2 + x_4) - \frac{\sqrt{2}}{12}(x_2 + x_4)^3 \quad , \tag{35}$$

where we have dropped the constant energy term. We obtain the coefficients λ_1, λ_2 and $H(\underline{\ell})$ by identification of (35) with (1). We see that, for each P_j, we arrive at the same prepared hamiltonian form (35) and hence we have a single MNF valid for all three P_j, as expected.

V. RESULTS

To present the results for the Hénon-Heiles hamiltonian, we will use the $\underline{z}$ coordinates of the last section. All numerical propagations were made via an optimized fourth order Runge-Kutta code [18] and, for the Poincaré sections, we use a trick by Hénon [19] for exact crossings of the surface sections.

First of all, we compute the coefficients of the special form $X(i, 0, 0, n_2, 0)$ and $X(i, 0, 0, 0, n_4)$. As we saw in section III, this computation is much easier than for the full set $X(i, \underline{n})$. So, with the same machinery and twice the real time for the general case,

we arrive only at the order 16 for the full set, against the order 50 for both subsets. The maximum order obtained for the coeficients $K(2\underline{m})$ was also 16.

Figures 3 and 4 show both the exact (un)stable manifolds that originate at the saddle point and their approximations via the coordinate transformations $\underline{z}(\underline{y})$ (see (31) and (5)), truncated at various orders up to the maximum of 50. We see that the exact curves meet smoothly in the bounded region. In fact, they pertain to a family of orbits entirely confined to the plane $z_1 = z_3 = 0$, as is easily seen from the form (34) of the Hénon-Heiles hamiltonian. To obtain the exact curves, we select one starting point near the origin at each axis y_2 or y_4 (see (17)) and then propagate them through the exact flow generated by $h(\underline{z})$. This method will be referred to in the following as *z-propagation*, being always precise, provided we start from $\underline{y}$ near the origin. The z-propagation is semianalytical in the sense that we numerically evolve precise initial conditions that are *analytically* given. We note that for localizing the "correct" initial directions i.e. those given by the axes y_2 and y_4, the form of (17) is crucial. On the other hand, the approximate curves in figures 3 and 4 are just the axes y_2 and y_4, merely rewritten in the coordinates $\underline{z}$ via the series truncated at different orders. This last method will be referred to in the following as *yz-translation* and its precision is only guaranted by Moser's theorem if $\underline{y}$ remains in the neighbourhood of the origin. The yz-translation is analytical in the sense that it uses only algebraic calculations and properties of the normal form.

The remarkable feature of both figures is that the analytic yz-translation clearly points to the convergence of the series (5) even *far from* the small region where Moser initially guaranteed it. This (and additional evidence in the following) strongly suggests the possibility of extending Moser's original convergence region. This issue is being considered in our present research. We also note that the region of accurate approximation ceases more and more abruptly as the truncation order become higher.

The next task is to obtain the unstable periodic orbits near the saddle point. They are entirely confined to the plane $y_2 = y_4 = 0$ ($\epsilon = 0$) of the normalized system, the family parameter being ρ (see (17)). We need in principle only the coefficients of the form

$X(i, n_1, n_3, 0, 0)$ to compute them. However, we saw in section III that those and all remaining coefficients must jointly be found. We obtained the full set $X(i, \underline{n})$ in this work until the order 16 and from now on this will be ever the case.

Figures 5 and 6 show the projections of three such unstable closed orbits in the planes $z_2 = z_4 = 0$ and $z_1 = z_3 = 0$, respectively. The values of ρ caracterizing each orbit are shown in the figures. For sufficiently small values of ρ we are always in the convergence region for $y_2 = y_4 = 0$ and both methods i.e. the z-propagation and yz-translation, generate closed curves which are indistinguishable in practice. Any orbit of the family closes once in figure 5 for every two returns in figure 6. We also note that the motion is microscopic or "residual" in the second figure (in the plane $z_1 = z_3 = 0$) relatively to the first one (in the plane $z_2 = z_4 = 0$). These features are easily understood if we consider $y_2 = y_4 = 0$ in (5) and (31) and keep only contributions from the $\ell = 2$ terms in (5).

We saw in section III that there exist a cylindrical flow associated to any of the unstable periodic orbits above. Four semicylinders emanate from each of them. Let τ be one such orbit and consider, for example, only the pair of (un)stable semicylinders which directly evolve from τ to the region $z_2 < 0$. This evolution occurs in an intrincate manner in the energy shell and, because the dynamics are chaotic, the semicylinders mutually intersect transversally. At each of these intersections there exists a homoclinic orbit (HO) which tends to τ as $t \to \pm\infty$.

We consider in this work only the first of the intersections above, in which we have the *primary* HOs. A suitable Poincaré surface to observe that is one that is transverse to the cylinders themselves, i.e., one that intersects each cylinder in an irreducible (closed) curve. The symmetry of figures 3 and 4 and the continuity of the cylinders with respect to the parameter ρ suggest that one such surface is the plane $z_4 = 0$. On the other hand, figure 6 shows that this plane also intersects the corresponding orbit τ (and hence the semicylinders themselves) in the region $z_2 > 0$. In fact, it is seen both qualitatively and numerically that in the region $z_2 > 0$ the cylinders do not mutually intersect at all, except obviously at τ itself. Moreover, the plane $z_4 = 0$ generates only reducible (open) curves on the cylinders in

that region i.e. it is longitudinal to the cylinders there. For sufficiently small ρ the orbit τ is near the origin, so we avoid numerical troubles, without losing precision, by starting on the cylinders at some small $z_2 < 0$.

We obtain the Poincaré section $z_4 = 0$ in the region $z_2 < 0$ by three methods. The first one is the z-propagation of the following initial conditions: a sufficiently small (constant) height $y_4 = y_{40}$ determines a straight section on the semicylinder characterized by $y_2 = 0$ (hence $\epsilon = 0$) and a given ρ (see (17)). Now, all orbits through the section are given by

$$\begin{cases} y_1 &= \rho \exp\left[i\theta\right] \\ y_3 &= -i\rho \exp\left[-i\theta\right] \quad , \quad \theta \in [0, 2\pi) \quad , \end{cases} \tag{36}$$

where the phase θ identifies each orbit on it. Obviously, we translate all initial points above from $\underline{y}$ to $\underline{z}$, before propagating them by the exact flow. The propagation of the other semicylinder, characterized by $y_4 = 0$ and the same ρ, is made in a similar way, except that the straight section is now determined by the small height $y_2 = y_{20}$. In all cases presented here it proved to be sufficient to use $y_{20} = y_{40} = 1.0 \times 10^{-2}$. We z-propagated both semicylinders until the condition $z_4 = 0$ was reached by the first time in the region $z_2 < 0$.

Figure 7 shows a typical Poincaré section as described above, namely that for $\rho = 1.0 \times 10^{-3}$. Four primary HOs are exhibited there. The highly symmetric features of this figure (and all succeeding ones) must be viewed as reflecting the intrinsic symmetries of the Hénon-Heiles potential. So, they do not depend on the specific value of ρ, which works somewhat as a scaling factor in all them. In particular, we see that the HOs are always on the axes, as indicated in figure 7. Each of these HOs emanates from the corresponding orbit τ along the unstable cylinder and returns to τ along the stable one. The remaining orbits do not exchange cylinders, at least at the first crossing. Figures 8, 9, 10 and 11 show, for $\rho = 1.0 \times 10^{-3}$, the evolution of a typical orbit on each cylinder until the Poincaré section was reached. On the other hand, figures 12, 13 and 14 show a primary HO associated to the orbit τ with $\rho = 2.6 \times 10^{-3}$, just that which corresponds to the HO localized on the left (i.e. at $z_3 < 0$) in figure 7. Figure 13 is an amplification of figure 12, revealing the microstructure

of the HO near τ. We note that two typical orbits on distinct semicylinders do not have the same coordinates when they reach, for the first time, the condition $z_4 = 0$, contrary to what occurs with the two branches of a primary homoclinic curve.

We also consider two analytical methods for the determination of the same Poincaré sections above. One of them is the yz-translation already presented. For this purpose we have merely to rewrite the condition $z_4 = 0$ in the $\underline{y}$ coordinates,

$$y_4^{N+1} + \sum_{n=2}^{N} [X(2, n_1, n_3, 0, n_4) - X(4, n_1, n_3, 0, n_4)] y_1^{n_1} y_3^{n_3} y_4^{N+n_4} = 0 \quad , \tag{37}$$

for the semicylinder for which $y_2 = 0$ and

$$y_2^{N+1} + \sum_{n=2}^{N} [X(2, n_1, n_3, n_2, 0) - X(4, n_1, n_3, n_2, 0)] y_1^{n_1} y_3^{n_3} y_2^{N+n_2} = 0 \quad , \tag{38}$$

for the one with $y_4 = 0$. Here, $N = 16$ and y_1 and y_3 are fixed through (36) for each value of θ. The equations (37) and (38) thus determine for each phase/orbit θ what is the corresponding height y_4 or y_2 where it crosses the plane $z_4 = 0$. As they are polynomials of degree $2N$ in y_4 and y_2 respectively, we have to select in each case, among the $2N$ complex roots, the only one that is *real* and verifies $z_2 < 0$. To find the roots, we use codes described in [18], based on the Laguerre method. Finally, we come back to the results in the $\underline{z}$ coordinates.

The second analytical method is essentially equivalent to the yz-translation and consists of evolving in time the solutions (17) themselves until the condition $z_4 = 0$ be verified. We will refer to it as the *y-propagation*. Even though the y-propagation be trivial, the comparison of both analytical methods is an important test for the numerical consistency of both sets of coefficients $X(i, \underline{n})$ and $K(2\underline{m})$ that we obtained, as the yz-translation involves *only* the coefficients $X(i, \underline{n})$. In fact, the figures for the Poincaré sections obtained through both analytical methods are indistinguishable in practice, which confirms the mutual consistency of those two sets of coefficients. On the other hand, in figure 15 we superimpose (for $\rho = 1.0 \times 10^{-3}$) the analytical and the semianalytical results, the latter already displayed in figure 7. We conclude that the analytical method is able to describe all qualitative aspects, such as symmetries, and the existence of the transverse cylindrical intersections and HOs

— and hence the chaotic nature of the motion. Moreover, we see that even its numerical efficiency is surprising if we realize that we are far from the region of guaranteed convergence. Considering that here the MNF has only been expanded to order 16, we are faced again with an indication that Moser's theorem can be extended.

Now we are ready to search for the closed orbits with long periods, which accumulate on the HOs. For this purpose, we have to consider the smooth selfintersecting cylinders for which not only $\rho \neq 0$ but also $\epsilon \neq 0$ (see section III). Due to the continuity of the cylindrical structure with respect to the parameters ρ and ϵ, we know that the transverse cylindrical selfcrossings must also occur at the plane $z_4 = 0$ in the region $z_2 < 0$. Then, all the previous discussion about both the semianalytical and analytical methods are still in order here, the only changes being now that the relation $y_2 y_4 = \epsilon$ eliminates both conditions (37) and (38) in favour of just one, e. g.

$$y_2^{N+1} - \epsilon\, y_2^{N-1} + \sum_{n=2}^{N} [X(2,\underline{n}) - X(4,\underline{n})] y_1^{n_1} y_3^{n_3} \epsilon^{n_4} y_2^{N+n_2-n_4} \;=\; 0 \quad , \tag{39}$$

with y_1 and y_3 still given by (36).

Figure 16 shows the Poincaré section above, obtained through the semianalytical method for the cylinder with parameters $\rho = \epsilon = 1.0 \times 10^{-3}$. It exhibits four orbits in the first autointersection of the cylinder, with the same symmetries already seen. Figures 17 and 18 show a typical orbit on that cylinder, from the time it crosses the plane $z_4 = 0$ through the branch of the cylinder which brings the orbit near to the origin (the *S-branch*), until the time it crosses that plane again, coming through the branch which takes it away from the origin (*U-branch*). We compared with each other the two analytical methods (yz-translation and y-propagation), confirming also in this case the underlying numerical consistency of both sets of coefficients $X(i, \underline{n})$ and $K(2\underline{m})$. In figure 19 we superimpose the analytical results with those displayed in figure 16.

What happens to the orbits in the selfintersections of figure 16? We easily discover by numerical computation that, they do not close in general for any specific values of ρ and ϵ — at least at the *first* intersection. In fact, as the S and U-branches of the cylinder

met smoothly near the origin, a more complex evolution can occur to *any* orbit before it eventually closes after a certain number of cylindrical selfintersections. We consider here the accumulation of periodic orbits only on the primary HOs, so we will limit the search of closed orbits just at the first crossing. For this reason, we fix ρ at some value and vary ϵ, thus verifying for each ϵ whether the four primary autointersections close on themselves or not. They in fact do so for a set of well defined discrete values of ϵ. In particular, the four curves close simultaneously on themselves when they do so, generating four distinct periodic orbits of the same period. We display in table I the numerical values of energy and period for the case $\rho = 2.6 \times 10^{-3}$. We present there only those orbits at the crossing localized on the left (with $z_3 < 0$) in figure 16, while the corresponding primary HO of interest lies at the same crossing, as in figure 7. PO1,...,PO4 are the periodic orbits we found, E is the energy and T is the period of each orbit. For the HO the finite numerical value for T is also shown. Due to the high numerical instability associated to the chaotic regime, the orbit are quickly lost after that time. We note the fast convergence of the values of E and also that the periods of the POs are (nearly) multiple of that of τ. On the other hand, the accumulation process of PO1,...,PO4 onto the corresponding HO, which in turn tends to τ as $t \to \pm\infty$, is clearly shown in figures 20, 21 and 22. These figures are typical of the behaviour of the periodic orbits for all other values of ρ.

Finally, an independent test was made to assure that the periodic orbits above are computed with precision by the present method, i.e. we ask whether the truncation at $N = 16$ of the series (5) is sufficient or not to yield precise results. For this sake we use a code [20] already known from the literature for computing periodic orbits in hamiltonian flows. By starting from a guessed periodic orbit, the code searches for an exactly closed orbit in its vicinity (in terms of either the energy or the period). That code is based on rewriting the linear finite difference integration of the flow in terms of the monodromy matrix and its main feature is the very fast convergence rate.

So, we enter as guessed periodic orbits, the closed ones we have obtained above and compare each of them with the corresponding exact curve we get through the code. It turns

out that both orbits are indistinguishable in all cases we considered. For example, the exact curves fit completely those shown in figures 20 to 22, at those scales. Hence, we conclude that the truncation at $N = 16$ we have used is sufficient to support the semianalytical results presented here.

VI. CONCLUSIONS

We have applied the theory of Moser's convergent normal forms to a two-degree of freedom hamiltonian flow near a saddle point. It turns out that the cylindrical topology of the flow is promptly revealed in the normal coordinates. The chaotic behaviour was studied directly in the continuous flow underlying the homoclinic tangle in a Poincaré section. We applied the method to the Hénon-Heiles hamiltonian. This allowed to compute structures related to chaotic motion in the full phase space rather than just a Poincareé section: the (un)stable manilfolds originated at the saddle point, the unstable periodic orbits with the cylinders they generate and the homoclinic orbits associated to them, and also the closed orbits with long periods which accumulate on the homoclinic ones.

We developed two methods for obtaining those structures. The first one is semianalytical in the sense that we numerically propagate initial conditions that we were able to find only through the normal form. The second method (in two versions — which are important to test the numerical consistency of all coefficients obtained) is entirely analytical, in the sense that it uses only algebraic calculations and properties of the normal form. The calculations are truncated at the order 16 and compared with a known numerical method for searching exact periodic orbits in a two-degree hamiltonian flow. Our semianalytical results proved accurate, thus showing the sufficiency of the truncation order that we have used. On the other hand, the analytical method was surprisingly good even far away from the saddle point and it was certainly able to exhibit all the qualitative aspects of the chaotic motion there.

It was also possible to compute analytically, as a particular case, the (un)stable manifolds originated at the saddle point up to the order 50, obtaining much better numerical agreement

for the analytical method. Both facts indicate the convergence of the normal form beyond the "microscopic" region originally established by the Moser's theorem. We are presently working out this extension.

Despite the mathematical interest in ascertaining the convergence of Moser's normal form as far as the first intersection of the stable and unstable manifolds, there is no doubt that, even for a simple polynomial potential as the Hénon-Heiles, the convergence will be very slow. In view of the huge difficulty in increasing the order of the normal form, it seems that the semianalytical method that we have used to obtain chaotic structures is to be preferred. This relies only on a relatively small truncation of the normal form near the saddle point, which produces accurate cylinders that may be extended by numerical integration.

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TABLES

	ϵ	E	T
PO1	5.836022×10^{-5}	0.1666200153	14.026
PO2	4.1216×10^{-8}	0.166678334123	21.381
PO3	2.9118×10^{-11}	0.166678375310	28.536
PO4	2.057124×10^{-14}	0.166678375339	35.791
HO	0	0.166678375339	(∞) 83.795
τ	0	0.166678375339	3.628

TABLE I

TABLE CAPTIONS

TABLE I. Parameters of some orbits discussed in the text with $\rho = 2.6 \times 10^{-3}$.

FIGURES

FIG. 1. Projections in the planes $y_2 = y_4 = 0$ (a) and $y_1 = y_3 = 0$ (b) of a hamiltonian flow near a saddle point.

FIG. 2. Equipotential curves of the Hénon-Heiles potential shown in eq. (27).

FIG. 3. Exact (un)stable manifolds that originate at the saddle point and their MNF approximations at various even truncation orders $2, \cdots, 50$. The exact curves meet smoothly at $z_4 = -1.5$ and a particle in the loop runs in the clockwise direction. Note the evidence of convergence of the MNF far from the origin.

FIG. 4. The same as in the figure 3 except that the truncations are made at the various *odd* orders shown. Note in this case the "out" direction of the divergence of the approximate curves from the exact one.

FIG. 5. Projection in the plane $z_2 = z_4 = 0$ of three unstable periodic orbits (for which $y_2 = y_4 = 0$ and hence $\epsilon = 0$) near the saddle point, for the values of ρ indicated in the figure.

FIG. 6. Projection in the plane $z_1 = z_3 = 0$ of the orbits shown in the figure 5. Note that two loops here corresponds to one loop in the other projection. Note also the microscopic motion here, relatively to that in the plane $z_2 = z_4 = 0$.

FIG. 7. Poincaré section at $z_4 = 0$ (in the region $z_2 < 0$) of the (un)stable semicylinders (for which $y_2 y_4 = \epsilon = 0$), originated at the orbit τ with $\rho = 1.0 \times 10^{-3}$. Note the four primary homoclinic orbits at the intersections on the axes.

FIG. 8. Projection of the (time-reversed) evolution of a typical orbit on the stable semicylinder, from a small initial condition already in the region $z_2 < 0$, until the condition $z_4 = 0$ was reached. The parameters are $\rho = 1.0 \times 10^{-3}$ and $y_2 y_4 = \epsilon = 0$.

FIG. 9. The other projection of the orbit shown in the figure 8.

FIG. 10. Evolution of a typical orbit on the unstable semicylinder in the same conditions of the figure 8.

FIG. 11. The other projection of the case shown in the figure 10.

FIG. 12. Projection of a primary homoclinic orbit (the HO indicated in the text), associated to the orbit τ with $\rho = 2.6 \times 10^{-3}$. The computed time was $T = 83.795$, much longer than the period $T = 3.628$ of τ, and its energy $E = 0.166678375339$ is the same of τ, as expected (see table I).

FIG. 13. Amplification of the figure 12 showing the microstructure of the HO near the orbit τ.

FIG. 14. The other projection of the homoclinic orbit shown in figure 12, compared with the corresponding orbit τ.

FIG. 15. Superimposition of the precise semianalytical results, already displayed in the figure 7, to that obtained by the analytical method. Note the qualitative agreement and even the surprising numerical proximity between them, although we are here very distant of the origin. This is an additional evidence for the convergence of the Moser normal form beyond that neighbourhood.

FIG. 16. Precise semianalytical obtaining of the Poincaré section at $z_4 = 0$ (in the region $z_2 < 0$) of the (smooth) cylinder for which $\rho = \epsilon = 1.0 \times 10^{-3}$. The resemblance with the figure 7 is due to the intrinsic simmetries of the Hénon-Heiles potential.

FIG. 17. Projection of a typical orbit on the (smooth) cylinder for which $\rho = \epsilon = 1.0 \times 10^{-3}$, from a small initial condition already in the region $z_2 < 0$, until the condition $z_4 = 0$ was reached in both time directions.

FIG. 18. The other projection of the orbit shown in the figure 17. Note that it closes on itself on the plane shown in that figure but not here.

FIG. 19. Superimposition of the results already displayed in the figure 16, to that obtained by the analytical method. The comments made in figure 15, about the comparison of the semianalytical and analytical methods, are in order here.

FIG. 20. Superimposition of the orbits of the table I, showing the accumulation of the periodic orbits (PO1 to PO4 in the table) onto the corresponding homoclinic orbit (HO), associated to the τ orbit with $\rho = 2.6 \times 10^{-3}$. There is no difference among them far from the origin at this scale.

FIG. 21. Amplification of the figure 20 near the origin. The accumulation process is much more slow here, on the contrary that occurs in the far distant region. The periodic orbits PO1 and PO2 of the table I does not reach this region.

FIG. 22. The same accumulation process of figure 20 projected now onto the other plane. These results are typical of all others values of ρ we have considered.